



## Power Spectrum Estimation of Randomly Sampled Signals

Velte, Clara M.; Buchhave, Preben; K. George , William

*Published in:*

Proceedings of the 17th International Symposium on Applications of Laser Techniques to Fluid Mechanics

*Publication date:*

2014

*Document Version*

Peer reviewed version

[Link back to DTU Orbit](#)

*Citation (APA):*

Velte, C. M., Buchhave, P., & K. George , W. (2014). Power Spectrum Estimation of Randomly Sampled Signals. In *Proceedings of the 17th International Symposium on Applications of Laser Techniques to Fluid Mechanics* (pp. 11)

---

### General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

## Power Spectrum Estimation of Randomly Sampled Signals

Clara M. Velte<sup>1,\*</sup>, Preben Buchhave<sup>2</sup>, William K. George<sup>3</sup>

1: Department of Mechanical Engineering, Technical University of Denmark, Kgs. Lyngby, Denmark

2: Intarsia Optics, Birkerød, Denmark

3: Department of Mechanical and Aerospace Engineering, Princeton University, Princeton NJ, USA

\* Correspondent author: cmve@dtu.dk

---

**Abstract** The random, but velocity dependent, sampling of the LDA presents non-trivial signal processing challenges due to the high velocity bias and the arbitrariness of particle path through the measuring volume, among other factors. To obtain the desired non-biased statistics, it has previously been shown analytically as well as empirically that residence time weighting is the suitable choice. Unfortunately, due to technical problems related to the processors providing erroneous measurements of the residence times, this previously widely accepted theory has been questioned and instead a wide spectrum of alternative methods attempting to produce correct power spectra have been invented and tested. The objective of the current study is to create a simple computer generated signal for baseline testing of residence time weighting and some of the most commonly proposed algorithms (or algorithms which most modern algorithms ultimately are based on), sample-and-hold and the direct spectral estimator without residence time weighting, and compare how they perform in relation to power spectra based on the equidistantly sampled reference signal. The computer generated signal is a Poisson process with a sample rate proportional to velocity magnitude that consist of well-defined frequency content, which makes bias easy to spot. The idea is that if the algorithms are not able to produce correct statistics from this simple signal, then they will certainly not be able to function well for a more complex measured LDA signal. This is, of course, true also for other methods that are based on the tested algorithms. The extremes are tested by increasing, e.g., the ‘turbulence intensity’ and the ‘shear’. It is observed that sample and hold and the free-running processor perform well only under very particular circumstances with high data rate and low inherent bias, respectively, while residence time weighting provides non-biased estimates regardless of setting. The free-running processor was also tested and compared to residence time weighting using actual LDA measurements in a turbulent round jet. Power spectra from measurements on the jet centerline and the outer part of the jet illustrate a distinct difference between the residence time weighted and the non-weighted spectra, in particular for positions far off the jet center axis where the bias increases.

---

### 1. Introduction

As shown previously (Buchhave 1979, Buchhave *et al.* 1979), the bias occurring in the velocity moments computed by standard methods applied to burst mode LDA data is removed when time averages are computed based only on the time the seed particles are actually present in the measurement volume, the so-called residence time. Residence time weighted (RTW) statistical moments were described theoretically in Buchhave (1979) and have been shown in Velte (2009), Buchhave *et al.* (2014) and Velte *et al.* (2014) to provide the expected results when applied to a turbulent free jet. However, the validity of the residence time weighted burst mode LDA processing is still disputed, partly by the promotion of alternative methods, partly with reference to practical problems with the correct measurement of the residence time (Albrecht *et al.* 2003). To illustrate the difference between the RTW method and conventional statistical computations we have created computer generated data sets for simple well-defined functions and compared the power spectra computed by different methods. In addition, we have computed power spectra obtained by LDA measurements in a turbulent jet. Our test signals are the following:

- A Gaussian pulse (Section 4)
- A flow velocity oscillating with a single frequency with variations in sample rate and mean value (Section 5).
- A flow velocity modulated with a superposition of five frequencies (Section 5).
- A Gaussian random signal (Section 6).
- A von Kármán model spectrum (Section 7).

## 2. Computer generated data

The simulated data is generated by first creating a high data rate primary velocity signal  $u(t)$ . This signal is then re-sampled by a Poisson process. The probability of the Poisson sampling (the Poisson sampling parameter) is modulated by the instantaneous velocity magnitude thereby introducing the velocity-sample rate correlation. The final data set consists of the measurement time (arrival time),  $t_k$ , the fluctuating velocity  $u'_k = u(t_k) - \bar{u}$ , where  $\bar{u}$  is the mean velocity computed by the relevant method (e.g. conventional or RTW), and the residence time  $\Delta t_k$ . Using these computer generated (CG) data, we present power spectra computed by a number of different algorithms:

- the residence time weighted direct spectral estimator
- the direct spectral estimator **without** residence time weighting (free-running processor, all  $\Delta t_k$  set to unity)
- the sample-and-hold method

Note that no dead time effects (Buchhave *et al.* 2014) have been intentionally included in the computer generated signals. In addition, we compare spectra computed by different methods on data measured in a turbulent free jet.

## 3. Algorithms and Residence Time Weighting

All power spectra are computed by performing averages of multiple realizations of the so-called direct method,

$$S(f) = \frac{1}{T} \mathcal{F}\{u'(t)\}^* \mathcal{F}\{u'(t)\} \quad (1)$$

where  $\mathcal{F}\{u'(t)\}$  is the Fourier transform of the velocity  $u'(t)$  based on a finite record length,  $T$ . The digital version of this formula is given by Buchhave (1979)

$$S_T(f) = \frac{T}{\left(\sum_{k=0}^{N-1} \Delta t_k\right)^2} \sum_{\substack{k=0 \\ k \neq k'}}^{M-1} \sum_{k'=0}^{N-1} e^{-i2\pi f(t_k - t_{k'})} u'_k \Delta t_k u'_{k'} \Delta t_{k'} \quad (2)$$

where  $S_T(f)$  is the spectral estimate for the finite record length  $T$ . The residence time weighting is automatically included with the residence times  $\Delta t_k$  and  $\Delta t_{k'}$  in equation (2), see e.g. Buchhave (1979) and Buchhave *et al.* (1979).

The primary advantage of the direct method is the fact that the power spectra based on random data computed by this method are in principle un-aliased, and that the method is fast when computed by array oriented software. The method results in a spectral offset, which must be subtracted to obtain the correct spectrum.

## 4. Gaussian Pulse

The Gaussian pulse is employed as a test case due to the broad spectrum that is produced by a short pulse of this type. The primary signal and its power spectrum are displayed in Figure 1. Note how the amplitude exceeds the mean value, so that the signal crosses zero velocity. The test case is interesting and difficult due

to the large differences in sample rate that occur due to the velocity modulation of the computer generated signal. The randomly sampled signal obtained from the reference and its sample-and-hold counterpart are shown in Figure 2. Note how the velocity modulation causes low data rates around zero ‘velocity’.

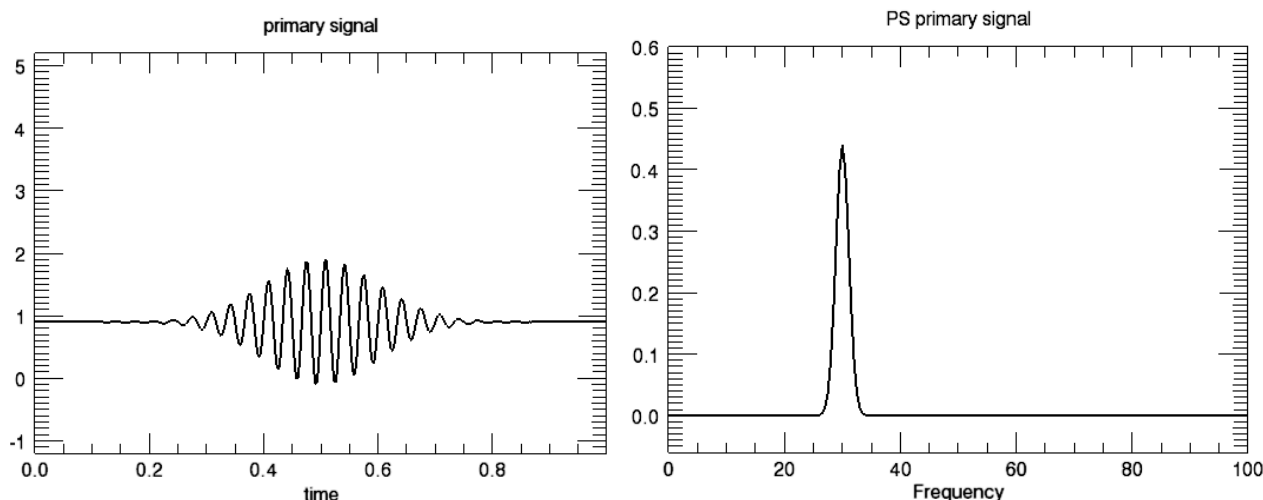


Fig. 1 Gaussian pulse reference signal (left) and corresponding power spectrum (right).

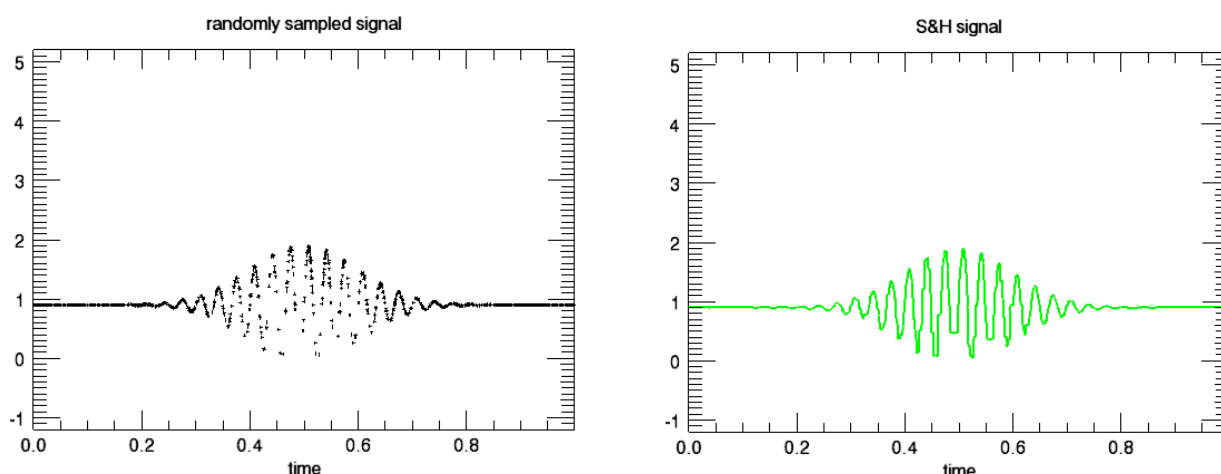


Fig. 2 Randomly sampled signal (left) and corresponding Sample-and-hold signal (right).

A comparison of the power spectra obtained by conventional averaging (left), residence time weighting (middle) and sample-and-hold (right) is given in Figure 3. The conventional spectrum shows erroneous power at low frequency and bias is further evident at the double frequency where an additional harmonic peak appears. The residence time weighted spectrum gives the correct spectrum, but the noise is higher than for the conventional spectrum. The sample-and-hold spectrum has lower power than the correct spectrum. Noise is low and there is no spectral offset.

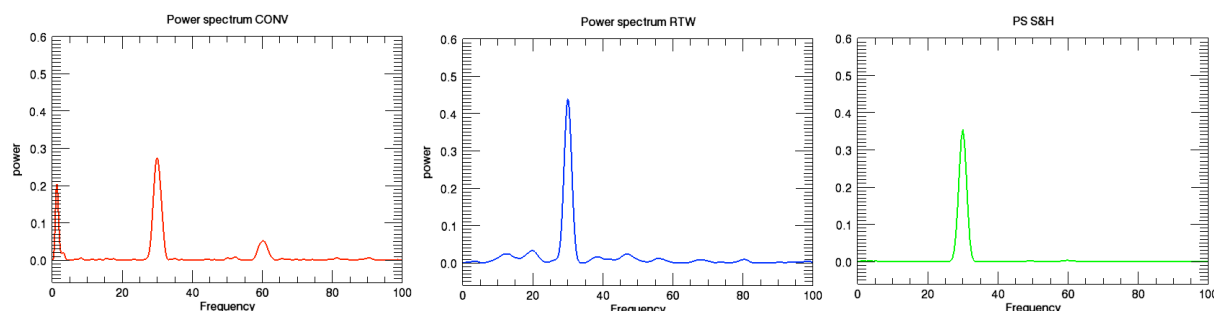


Fig. 3 Power spectra obtained using conventional averaging (left), residence time weighting (middle) and sample-and-hold (right).

## 5. Sine-wave signals

A longer pulse with a narrower spectrum can be obtained by studying a sine-wave. Consider first the single frequency modulated velocity signal. The primary signal and its corresponding power spectrum are displayed in Figure 4. Note how the primary signal has its one extreme overlapping zero value. This corresponds to large variations in turbulence intensity and shear. This setting is interesting since it results in large variations in sample rate and therefore also bias. The wiggles around the base of the power spectrum are the effect of the rectangular window (the sinc-squared frequency window), which shows up also with random sampling. This frequency window also determines the frequency content of the noise (compare e.g. the RTW spectrum from the short Gaussian pulse and the continuous sine wave with a rectangular window). The randomly sampled signal obtained from the reference signal and its sample-and-hold counterpart are shown in Figure 5.

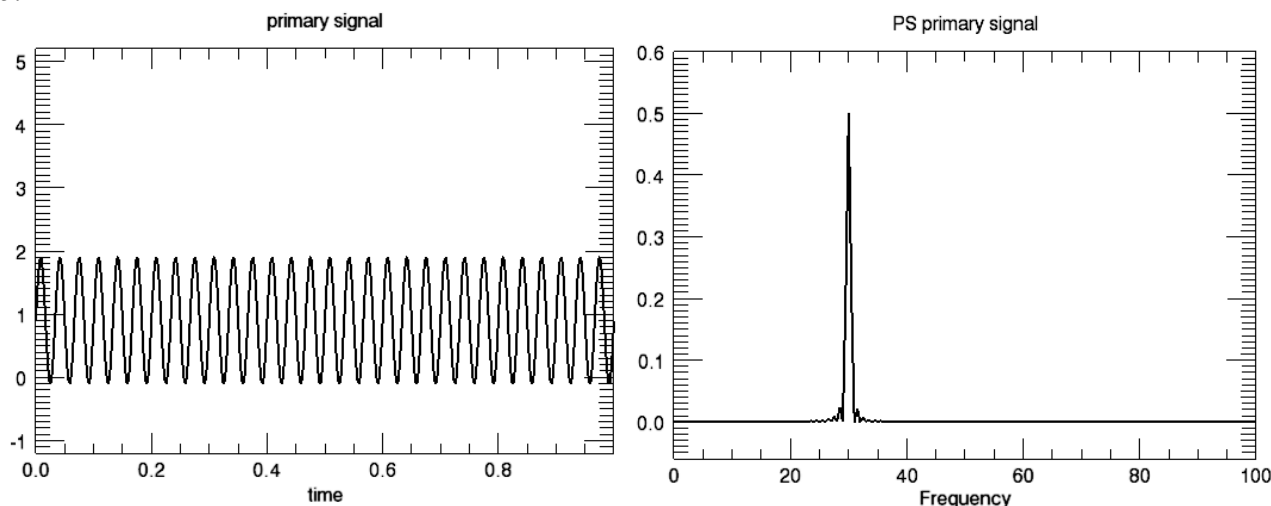


Fig. 4 Primary signal (left) and corresponding power spectrum (right).

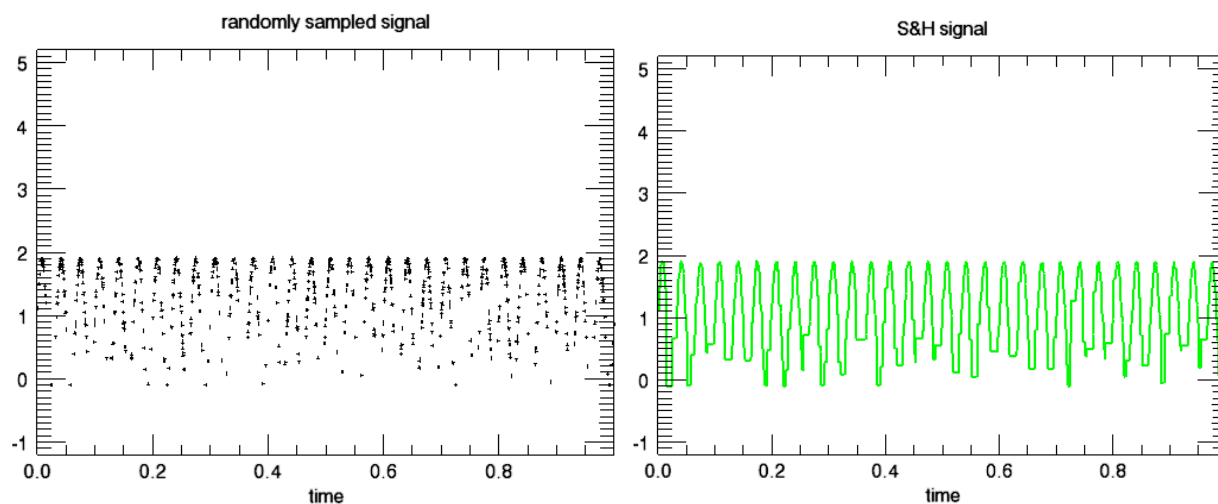
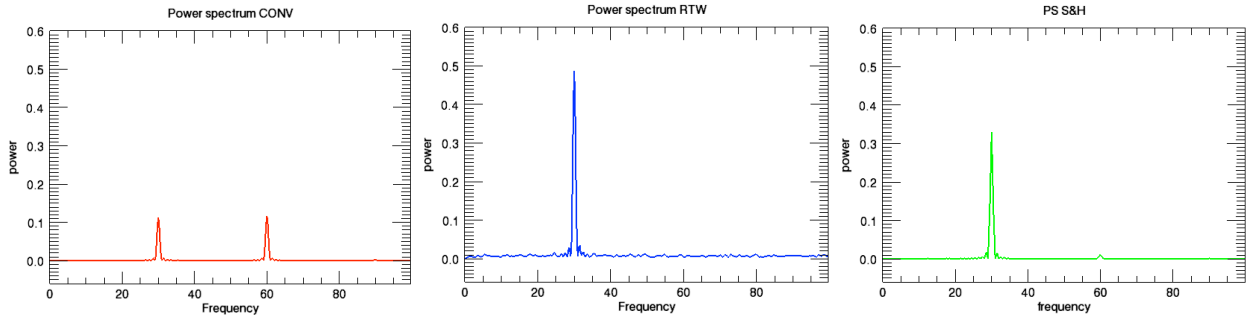


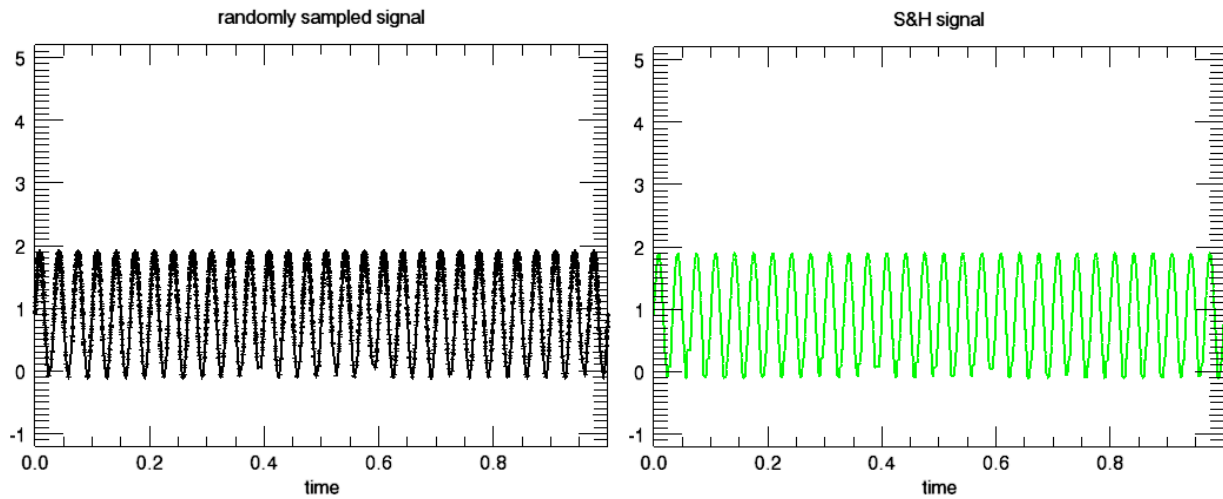
Fig. 5 Randomly sampled signal (left) and corresponding Sample-and-hold signal (right).

The power spectra obtained with each respective method from the randomly sampled signal is shown in Figure 6. The conventional spectrum shows erroneous power at the double frequency and peak value is much too low. The RTW spectrum is still correct and the wiggles from the window are clearly visible and they should be. The S&H spectrum has lower power than the correct spectrum. The noise level is low and there is no spectral offset. The next harmonic at 60 Hz is also visible.



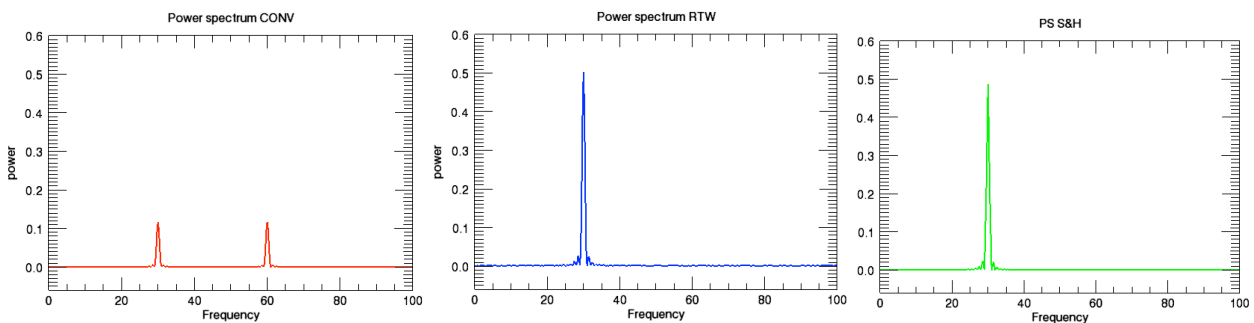
**Fig. 6** Power spectra obtained using conventional averaging (left), residence time weighting (middle) and sample-and-hold (right).

To test the effect of sample rate, we have simulated the same signal as above, but with 10 times higher average sample rate. This should reduce noise and be beneficial in particular for the sampled and held signal, which will of course be more representative of the primary signal. The velocity bias is still present in the signal, so the conventional direct estimator is not expected to improve with increased sample rate. The randomly sampled and corresponding sampled-and-held velocity signals are shown in Figure 7.



**Fig. 7** Randomly sampled signal (left) and corresponding Sample-and-hold signal (right) for 10 times higher average data rate.

The corresponding spectral estimates are shown in Figure 8. The conventional spectrum still shows erroneous power at the double frequency. The RTW spectrum is correct. The S&H spectrum has a bit lower power than the correct spectrum, but it has become much better with the high sample rate, and there is no velocity bias effect at higher harmonics.



**Fig. 8** Power spectra obtained using conventional averaging (left), residence time weighting (middle) and sample-and-hold (right) for 10 times higher average data rate.

Now to test the bias effect, let us again use the original signal and only change the mean value of the reference signal. The reference signal and the corresponding power spectrum are displayed in Figure 9. Since the signal is not oscillating around zero value anymore, the higher mean value should reduce velocity bias effect and also make the S&H spectrum better. The randomly sampled signal and its sampled-and-held counterpart are shown in Figure 10.

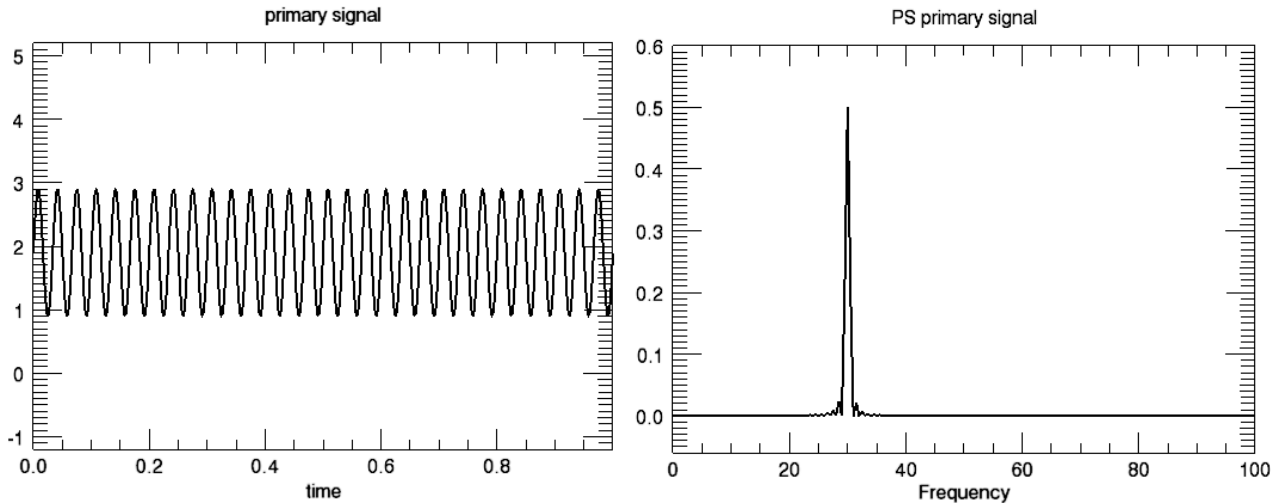


Fig. 9 Primary signal (left) and corresponding power spectrum (right) with a higher mean value.

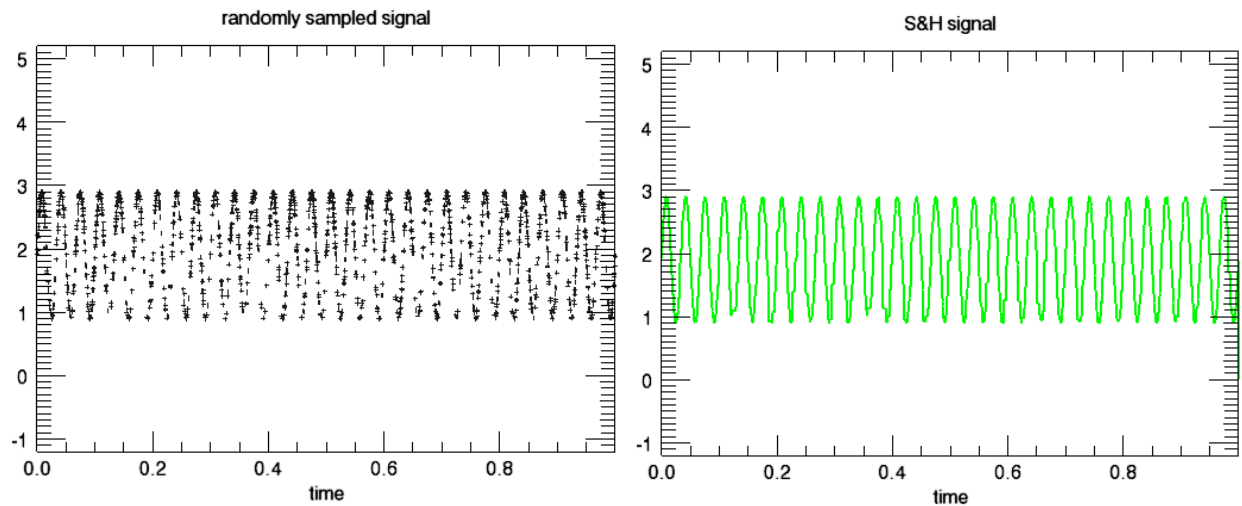


Fig. 10 Randomly sampled signal (left) and corresponding Sample-and-hold signal (right).

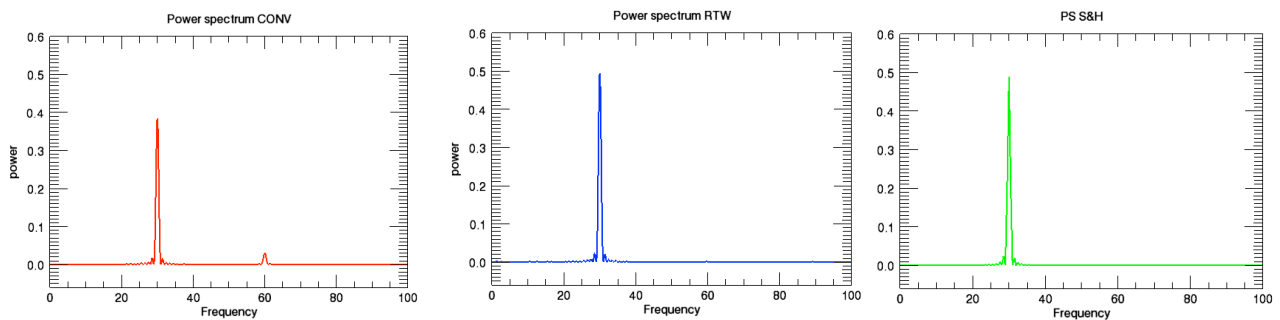
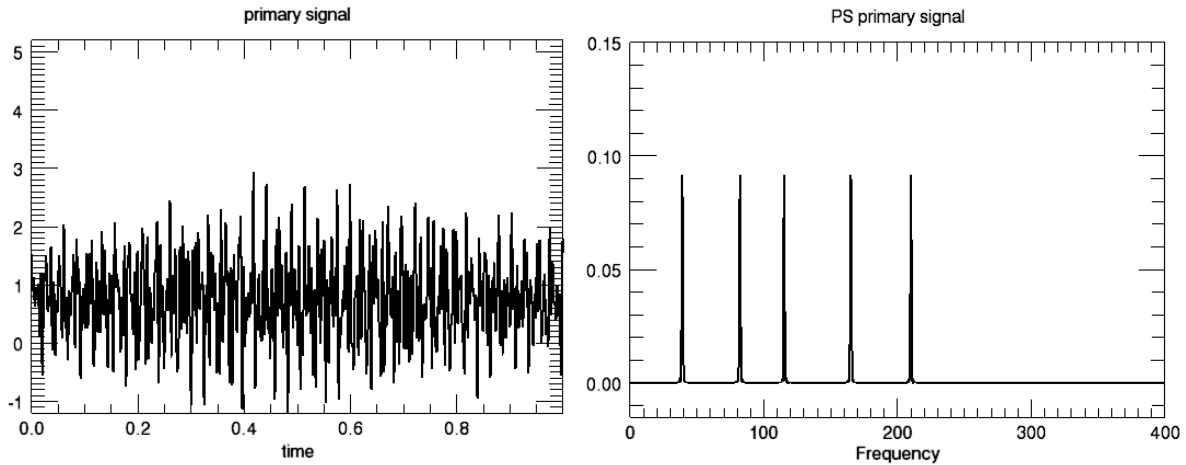


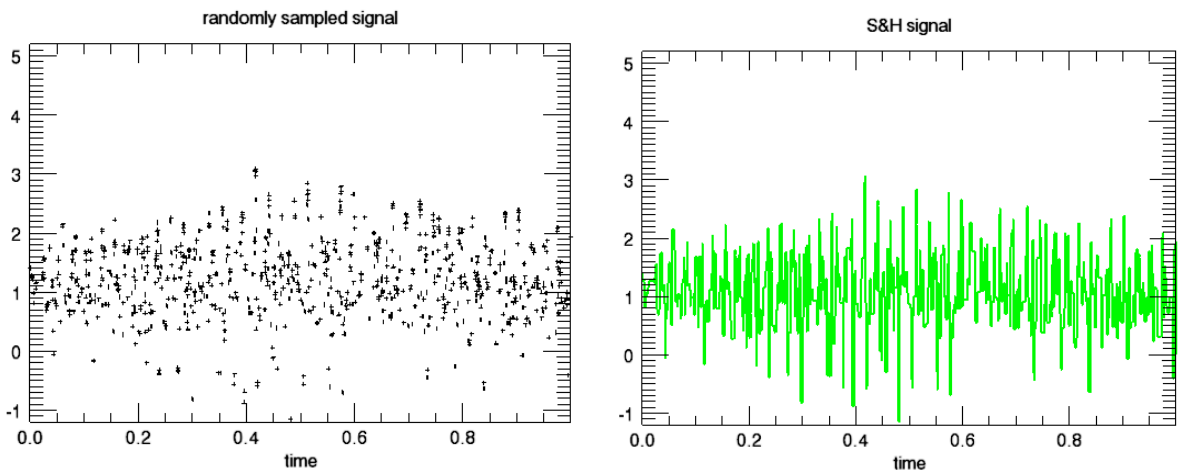
Fig. 11 Power spectra obtained using conventional averaging (left), residence time weighting (middle) and sample-and-hold (right) with a higher mean value of the reference signal.

Figure 11 displays the corresponding spectral estimates. The conventional spectrum looks much better, as expected for this lower value of ‘turbulence intensity’, though it still shows bias in the form of erroneous power at the double frequency and the main peak is too low. The RTW spectrum is correct. The S&H spectrum has a bit lower power than the correct spectrum, but there is no velocity bias effect in higher harmonics.

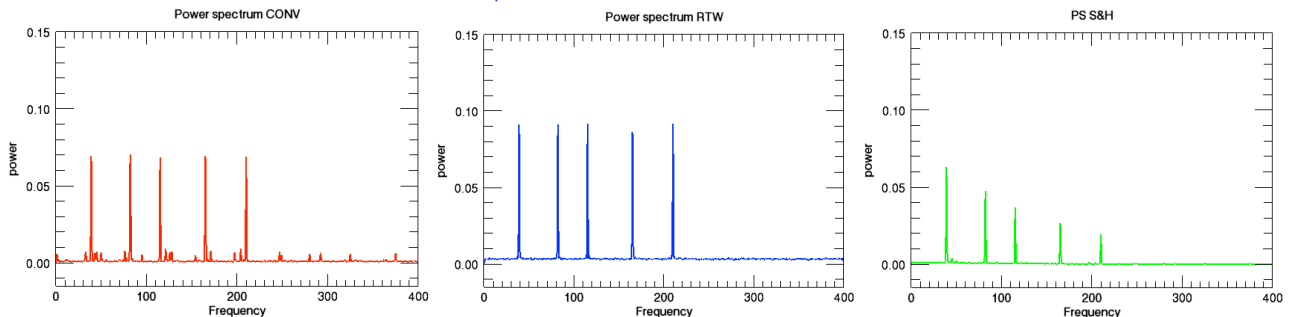
For completion, let us investigate a signal with a sum of five sine waves with different frequencies and random phases, see Figure 12. The randomly sampled signal and the corresponding sampled-and-held signals are displayed in Figure 13. The same data rate has been used as the original signal in Figure 5.



**Fig. 12** Primary signal (left) and corresponding power spectrum (right) with a sum of five sine-waves.



**Fig. 13** Randomly sampled signal (left) and corresponding Sample-and-hold signal (right).



**Fig. 14** Power spectra obtained using conventional averaging (left), residence time weighting (middle) and sample-and-hold (right).



Figure 14 shows the spectral estimates computed from these data sets. The conventional spectrum shows multiple erroneous mixing frequencies and the peak values are too low. The RTW spectrum is correct. The S&H spectrum has lower power than the correct spectrum, and the  $f^{-2}$ -filtering effect that appears for low enough data densities is evident, see Adrian and Yao (1987).

## 6. Gaussian Random Signal

For the Gaussian random signal, we have constructed three different test cases with varying mean value and amplitude, see Figure 15. The non-weighted direct Fourier transform (green) displays increased bias with increased ‘turbulence intensity’, in agreement with the high-velocity bias. Note again that the residence time weighted algorithm (blue) agrees well with the reference (red) in all cases.

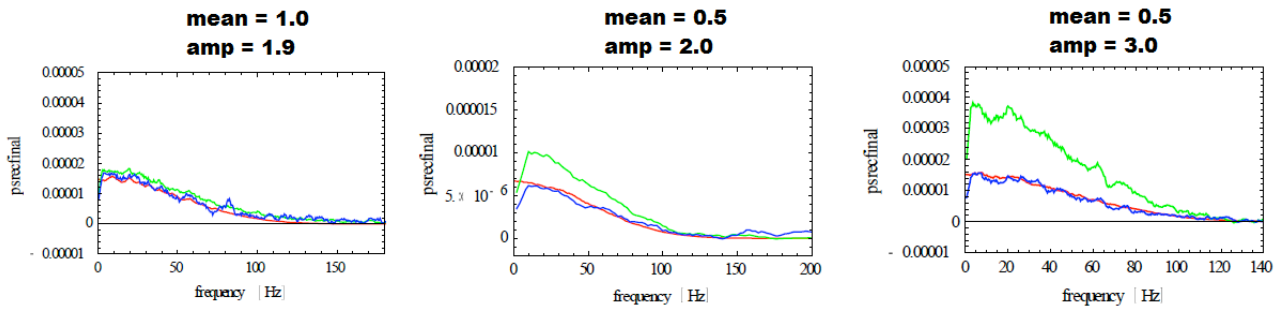


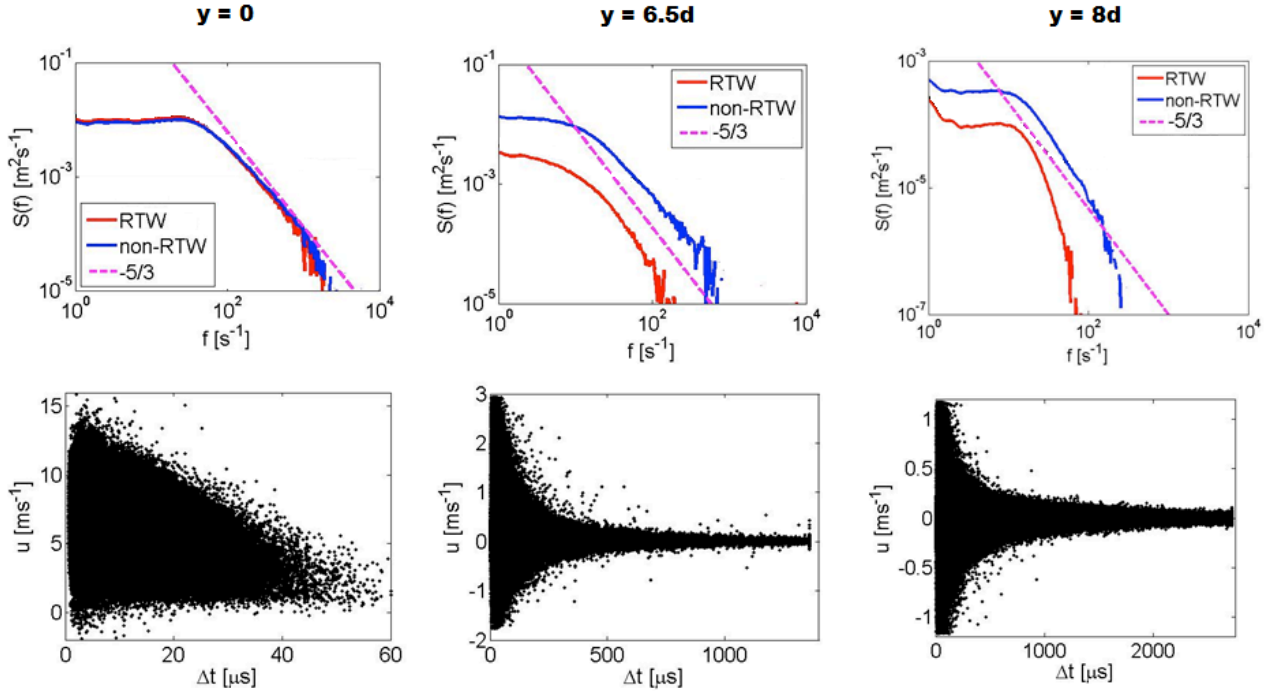
Fig. 15 Gaussian random signal. Red – reference, blue – residence time weighted, green – direct Fourier transform **without** residence time weighting.

## 7. Measured LDA Power Spectra

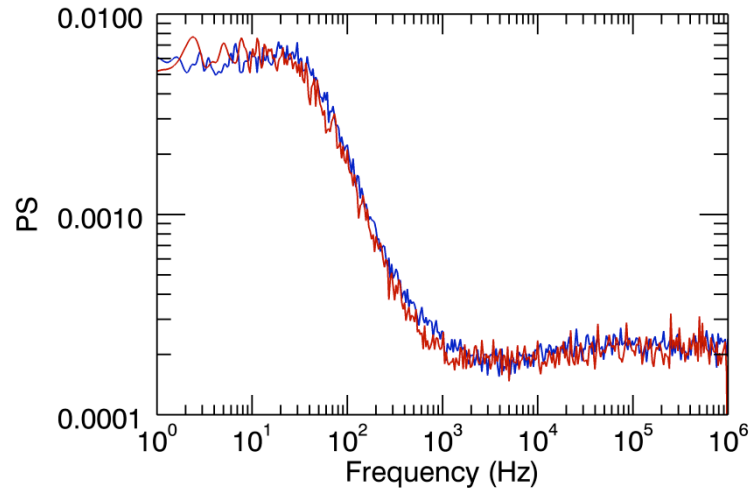
We further confirm the RTW algorithm by evaluating power spectra from a turbulent jet (see Velte 2009), both at the center axis and off the center axis 30 jet exit diameters downstream with the jet running at 30 m/s, see Figure 16. The spectra presented are (black) the residence time weighted one, (red) the same but corrected for the white noise floor, (blue) the direct spectral estimator **without** residence time weighting (all  $\Delta t_k$  set to unity), which is also corrected for white noise. The purple broken line shows the  $-5/3$  slope, but does not indicate any expected power law behavior. It is clear that as one moves away from the jet centerline, the discrepancy between the red and the blue curves increases. The instantaneous velocity/residence times scatter plots below the spectra show that for 6.5 and 8 diameters off the axis, the velocities are centered around zero velocity and the residence times increase dramatically around zero velocity, as expected. At these positions, in the outskirts of the jet, where low velocities, large residence times and high turbulence intensities occur, the bias grows dramatically and the residence time weighting becomes crucial for a correct representation of the power spectra. For validation, one can compare the integral of the power spectrum with the variance of the corresponding velocity data, which should be equal by definition. (But note that this will be true even if there is spectral leakage due to windowing, since the part of the spectrum ‘leaked’ to higher frequencies preserves the variance.) This was done by Velte *et al.* (2014) for the data of the same measurement campaign, who showed that residence time weighting is the only weighting that produces correct results.

Finally, we process the computer generated (CG) data and the measured velocity data through the same spectral estimator. As the real measurement volume diameter is a quantity that depends on a number of parameters such as particle size, detector/amplifier gain etc. we have adjusted the model measurement diameter  $d_{MV}$  to give the best fit to the measured turbulence spectrum. The measurement volume diameter affects the width and location of the dip in the spectrum. Even with this adjustment, the offset level of the computer generated spectrum is lower than that of the measured spectrum, even with approximately the same data rate. We therefore add random white noise in the frequency domain before the frequencies are converted to a time series. Such noise may be detector shot noise, thermal noise in electronics or phase noise

in the detected Doppler signal. Addition of this noise raises the constant noise floor. Finally, we add a small amount of fixed dead time ( $\approx 4 \mu\text{s}$ ) to the residence time distribution, see Buchhave *et al.* (2014). This additional dead time could be caused by a small finite processing or data transfer time added to the measured residence time. The two curves, the measured turbulence spectrum and the computer generated spectrum, now show excellent agreement, see Figure 17.



**Fig. 16** Stream-wise burst-mode LDA velocity spectra and corresponding instantaneous velocity/residence time scatter plots from an axi-symmetric turbulent jet at (first column) jet center axis, (middle column) 6.5 jet exit diameters off the center axis and (right column) 8 jet exit diameters off the center axis.



**Fig. 17** The measured turbulence spectrum (blue) and the CG spectrum with the measured Weibull residence time distribution plus a small fixed dead time (red). A constant noise level has been added to the computer generated von Karman spectrum.

## 8. Summary and conclusions

A non-biased representation of the frequency content of the LDA signal by residence time weighting has been validated by simple computer generated signals with different well-defined frequency content. The

computer generated signals are Poisson processes with a modulation to create the velocity – data rate correlation necessary to introduce high velocity bias which is typical of LDA signals. The resulting randomly sampled signal is less complex than a measured LDA signal, so any algorithm that cannot correctly predict the spectra from these signals cannot be expected to correctly predict the spectra from real LDA measurements in general. This is naturally also true for any method that is based on the tested failing algorithms. For completeness the effect of bias was also investigated in real LDA measurements in a turbulent axisymmetric jet, both at the centerline and in the outer parts where the bias is expected to increase. Since a hot-wire cannot accurately measure the flow at these off-axis positions, it is vital to be able to interpret the burst-mode LDA signal correctly to obtain non-biased results.

The test cases have been chosen due to the high degree of bias and therefore their ability to properly test the effect of the tested algorithms. As is clearly seen from the spectra of the Gaussian pulse, the single and sum of five sine waves, the distortion in the measured signal caused by the velocity – data rate correlation results in the generation of higher harmonics for S&H and the free running processor, most pronounced when the degree of modulation is high (equivalent to high ‘turbulence intensity’). The higher degree of modulation corresponds, e.g., to the conditions off the center axis and in the outer layer of a free jet or in a turbulent boundary layer in real flows. Simultaneously with the generation of higher harmonics, the fundamental is reduced. The RTW results show no sign of higher harmonics and the primary peak is well predicted. The S&H spectrum is better represented at high average data densities, where the sampled-and-held signal is more representative of the reference signal, but fails as expected at low data rates.

For the Gaussian random signal and the measured power spectra from a turbulent axisymmetric jet it is clear that the increase in shear and turbulence intensity produces increased discrepancy between the RTW and the non-weighted algorithms. The correctness of the RTW algorithm can be validated by simply comparing the signal variance to the integral of the power spectrum which should, by definition, be equal. Further, our simple model for mimicking LDA data and the dead time effects of the burst processor showed excellent agreement with the measured LDA power spectra. This and therefore provide a good

One of the main purposes of the current study is to highlight the necessity to test competing LDA signal processing algorithms in flows where there actually exists bias. In the vast majority of studies presented, the algorithms are tested in flows where the bias is small or negligible (no shear, low turbulence intensity, etc.). A more proper test could be, e.g., in a boundary layer or in the shear layer of a turbulent jet. As a final check for power spectra, the integral of the spectrum should always be compared (and be equal) to the signal variance. But note that this alone is not enough, since window leakage preserves the variance even though it distorts the spectrum.

## References

- Adrian RJ, Yao CS (1987) Power spectra of fluid velocities measured by laser Doppler velocimetry, *Exp Fluids* **5**:17-28.
- Albrecht H-E, Borys M, Damaschke N, Tropea C (2003) *Laser Doppler and phase Doppler Measurement Techniques*, Springer Verlag, pp. 552.
- Buchhave P (1979) *Errors and correction methods in Turbulence Measurements with the LDA*, Ph.D dissertation, State University of New York at Buffalo.
- Buchhave P, George WK, Lumley JL (1979) The measurement of turbulence with the laser-Doppler anemometer, *Ann Rev Fluid Mech* **11**:443-504.
- Buchhave P, Velte CM, George WK (2014) The Effect of Dead Time on Randomly Sampled Power Spectral Estimates, *Exp Fluids* **55**:1680 DOI 10.1007/s00348-014-1680-1.

Velte CM (2009) Characterization of vortex generator induced flow, PhD dissertation, Technical University of Denmark.

Velte CM, George WK, Buchhave P (2014) Estimation of Burst-Mode LDA Power Spectra, Exp Fluids 55:1674 DOI 10.1007/s00348-014-1674-z.